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IRREVERSIBLE INVESTMENT DECISIONS UNDER RETURN AND TIME UNCERTAINTY: OPTIMAL TIMING WITH A POISSON CLOCK

JUKKA LEMPA

Abstract. We study optimal timing of irreversible investment decisions under return and time uncertainty. The considered models are formulated as maximization problems of the expected present value of the exercise payoff, where the underlying dynamics follow a diffusion process. We formulate and study three variants of the benchmark model, namely the classical perpetual problem á-la Samuelson-McKean. Into each of these variants, we incorporate a different type of time uncertainty in terms of an exogenous Poissonian noise. For each variant, we propose a set of assumptions on the underlying and the payoff structure under which we can solve the timing problem. Furthermore, we study the interrelations of the timing problems and their interpretations. Finally, the results are illustrated with an explicit example.

1. Introduction

In many economical and financial applications, timing of an irreversible investment decision has a central role. A popular way of modeling such timing problem is to use a real option approach. In this approach, the timing problem is formulated analogously to the exercise timing of a financial option, which, in turn, is in many cases closely related to optimal stopping problems, where the object is the expected present value of the total return from the investment project, see, e.g., [19], [27], [33], and [34], see also [16] and [10] for textbooks on real options and irreversible investment. The purpose of this paper is to discuss and study four different classes of optimal stopping problems where the underlying dynamics follow a diffusion process. As a benchmark, we use the classical perpetual problem á-la Samuelson-McKean (SMcK), see [23]. This problem has been studied extensively under various degrees of generality, see, e.g., [2], [15], [22], [31], see also [28] for an up-to-date textbook on optimal stopping. From economic point of view, this model is built on a number

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Address. Jukka Lempa, Centre of Mathematics for Applications, University of Oslo, PO Box 1053 Blindern, NO – 0316 Oslo, Tel.: +47 22 85 77 04, Fax: +47 22 85 43 49, e-mail: jlempa@cma.uio.no.

of assumptions. In particular, it assumes that the decision maker is free to choose the exercise time based only on the complete information on the underlying without any exogenous restriction. Moreover, it assumes that the exercise payoff is settled immediately, and that the opportunity to exercise does not default, i.e. the time horizon is infinite. To discuss these assumptions, we formulate three variants of the SMcK-model, each with different type of time uncertainty dictated by a Poisson clock, that is an independent Poisson process.

As the first modification, we consider a version of SMcK-model where the underlying diffusion and the exogenous Poisson process are both started at the initial time. Moreover, we assume that the decision maker observes both processes continuously in time, and that the set of admissible exercise times is restricted to the jump times of the Poisson process. One interpretation of this constraint is liquidity effect, namely that, the opportunities to exercise do not occur continuously in time, but they are triggered by an exogenous factor modeled by the jumps of the Poisson process; for related studies in consumption/investment optimization, see, e.g., [30] and [26], in optimal stopping, see [17], [18], and [21], and in bounded variation control, see [32]. As an example, consider the timing of liquidation of a firm operating in traditional, product oriented industry. In such case, there can be a considerable amount of capital sunk in a possibly highly specialized production machinery. In order to liquidate the firm, the machinery must first be realized, which can take time due to potentially sparse demand. In other words, the market for such machinery can be highly illiquid. Now, the jump times of the Poisson process mark the moments when there is demand on the market.

In the second version of SMcK-model we consider optimal timing with exercise lag. Now, the decision maker observes the underlying diffusion continuously in time starting at the initial time. Moreover, she is free to choose the exercise time based on the information on the underlying without any exogenous restriction. However, upon exercise the exogenous Poisson process is started and the payoff is settled at the first jump time of this process. This model is closely related to studies with "delivery lag" or "time to build"-models, see, e.g., [4], [5], [6], [14], and [24]. Generally speaking, a distinctive feature of these studies is that the flow of revenue from investment starts after a delivery or building period, which can be either deterministic or random. As an example, consider again the particular type of liquidation problem from the previous paragraph. The model with delivery lag corresponds now to the case where after the liquidation decision, there is an exponentially distributed waiting time after which the sunk capital is realized. We remark that there is a fundamental difference between the first two modifications. Indeed, in the first version the market is irresponsive to the signals given by the decision maker in the sense that the opportunities to realize the machinery appear independently of her action, whereas in the second modification her actions affect the expected waiting time to have the demand.

Third modification of the benchmark exhibits a random expiry of the exercise opportunity. Similarly to the first modification, the underlying diffusion and the exogenous Poisson process are both started at the initial time, and decision maker observes both processes continuously in time. Again, she is able to choose the exercise time based on the information on the underlying. However, the exercise opportunity expires at the first jump time of the Poisson process making any further payoffs vanish. This leads into a random time horizon formulation of the optimal timing model, see, e.g., [12], and [13]. For related studies in asset pricing and optimal consumption/investment modeling, see [7] and [8]. As an example, consider again the previous liquidation problem where a new production technology, which is cheaper and more efficient, is introduced to the industry after an exponentially distributed waiting time. This will make the disinvestment option worthless or at least lower its value dramatically as the resale value of the old machinery collapses.

The reminder of the paper is organized as follows. In Section 2 we set up the underlying dynamics. In Section 3 we formulate the optimal timing problems, and prove and discuss the main results. In Section 4 we illustrate our results with an explicit example and Section 5 concludes the study.

2. The Underlying Dynamics

Before going into the formal description of the optimal timing problems, we set up the underlying dynamics. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, be a complete filtered probability space satisfying the usual conditions, see [9], p. 2. We assume that the state process X is defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, evolves on $\mathbf{R}_+$, and follows the regular linear diffusion given as the weakly unique solution of the Itô equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x,$$

where the functions μ and $\sigma > 0$ are sufficiently well behaving (say, continuous). Here, W is a Wiener process. In line with most economical and financial applications, we assume that the upper boundary ∞ is natural and that X does not die inside the state space $\mathbf{R}_+$, i.e., that killing of X is possible only at origin. Therefore the boundary 0 is either natural, entrance, exit or regular. In the case the origin is regular, it is assumed to be either killing or reflecting, see [9], pp. 18–20, for a characterization of the boundary behavior of diffusions. As usually, we denote as $\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ the second order linear differential operator associated to X. Furthermore, we denote as, respectively, ψ_r and φ_r the increasing and decreasing solution of the ODE $\mathcal{A}u = ru$, where $r > 0$, defined on the domain of the characteristic operator of X – for a characterization and the fundamental properties of the minimal r-excessive functions ψ_r and φ_r, see [9], pp. 18–20. Finally, we define the speed measure m and scale function S of X via the formulaæ $m'(x) = \frac{2}{\sigma^2(x)}e^{B(x)}$ and $S'(x) = e^{-B(x)}$ for all $x \in \mathbf{R}_+$, where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$, see [9], pp. 17.

We assume that the filtration $\mathbb{F}$ carries also the information of an independent Poisson process N with rate $\lambda > 0$. The process N jumps at times $T_1 < T_2 < \cdots < T_n < \ldots$, where the intervals $\{T_1, T_2 - T_1, T_3 - T_2, \ldots\}$ are exponential IID with mean $\lambda^{-1} < \infty$. By convention, we set $T_\infty = \infty$. Finally, we denote as $\mathbb{G} = \{\mathcal{G}_t\}_{t \geq 0}$ the natural filtration generated by the underlying X.

For $r > 0$, we denote as L_1^r the class of real valued measurable functions f on $\mathbf{R}_+$ satisfying the condition $\mathbf{E}_x \left[\int_0^\zeta e^{-rt} |f(X_t)| \, dt \right] < \infty$ for all $x \in \mathbf{R}_+$, where $\zeta = \inf\{t \geq 0 : X_t \leq 0\}$. In economical literature, the L_1^r-condition is typically referred to as *absence of speculative bubbles*, see [16]. For a function $f \in L_1^r$, the *resolvent* $R_r f : \mathbf{R}_+ \to \mathbf{R}$ is defined as

$$(R_r f)(x) = \mathbf{E}_x \left[\int_0^\zeta e^{-rs} f(X_s) ds \right], \tag{1}$$

for all $x \in \mathbf{R}_+$. The resolvent R_r and the functions ψ_r and φ_r are connected in a very convenient way. Indeed, we know from the literature that for given $f \in L_1^r$ the resolvent $R_r f$ can be expressed as

$$(R_r f)(x) = B_r^{-1} \varphi_r(x) \int_0^x \psi_r(y) f(y) m'(y) dy + B_r^{-1} \psi_r(x) \int_x^\infty \varphi_r(y) f(y) m'(y) dy, \tag{2}$$

for all $x \in \mathbf{R}_+$, where $B_r = \frac{\psi_r'(x)}{S'(x)} \varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)} \psi_r(x)$ denotes the Wronskian determinant, see [9], pp. 19. Finally, we remark that the value of B_r does not depend on the state variable x but on the rate r.

3. The Optimal Timing Problems

In the previous section we set up the underlying dynamics. Having done that, we formulate now the optimal timing problems described in the introductory section. Before going into formal description, we state the standing assumptions under which the first three problems are studied.

Assumption 3.1. *We assume that*

- *The payoff $g \in L_1^r$ is non-negative, continuous and nondecreasing*
- *There is a unique state x_1^* which maximizes the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ and that this function is non-decreasing on $(0, x_1^*)$ and non-increasing on (x_1^*, ∞) and that the limiting conditions $\lim_{x \to 0+} \frac{g(x)}{\psi_r(x)} = \lim_{x \to \infty} \frac{g(x)}{\psi_r(x)} = 0$ hold.*

We make some remarks on Assumption 3.1. From the application point of view, the L_1-condition is not particularly restricting. It states that the total cumulative expected present value of the cash flow $g(X_s)$ must be finite. As was mentioned in the previous section, the function ψ_r can be identified as an increasing solution of the ordinary second order differential equation $(\mathcal{A} - r)\psi_r = 0$ satisfying suitable boundary conditions. Even though it is not possible solve ψ_r from this ODE explicitly except in special cases, there are well developed methods for solving such equations numerically, cf. [1] and [35]. These methods can applied to the numerical verification of the assumed monotonicity and limiting conditions of the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ for a particular model specification.

3.1. **Timing problem #1: Classical case.** As was mentioned in the introduction, the optimal timing problems considered in this study are maximization problems of the expected present value of the exercise payoff. We assume that in every problem the underlying dynamics follow the linear diffusion X described in Section 2. Furthermore, in first three problems we assume that the exercise payoff is given by the payoff function $g : \mathbf{R}_+ \to \mathbf{R}$ satisfying Assumption 3.1. As a reference model, we use the classical perpetual problem á-la Samuelson-McKean, where the optimization is done over the entire set of $\mathbb{G}$-stopping times – recall the definition of $\mathbb{G}$ from previous section. Informally speaking, the decision maker can choose the exercise time freely based on the total information on the underlying X up to a given time – and only on that. The optimal timing problem is formulated as

$$V_1(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau) \right], \tag{3}$$

where the constant $r > 0$ is the discount rate and τ varies over all $\mathbb{G}$-stopping times. Next result characterizes the optimal exercise rule and the optimal value for Problem (3) under Assumption 3.1.

Theorem 3.2. *Let Assumption 3.1 hold. Then the global maximum* $x_1^* = \operatorname{argmax}\left\{\frac{g(x)}{\psi_r(x)}\right\}$ *constitutes the optimal exercise rule for the optimal timing problem* (3)*. Moreover, the value function* V_1 *can be rewritten as*

$$V_1(x) = \begin{cases} g(x), & x \geq x_1^*, \\ \frac{g(x_1^*)}{\psi_r(x_1^*)}\psi_r(x), & x < x_1^*. \end{cases} \tag{4}$$

Proof. See, e.g., [2], Theorem 3. □

Theorem 3.2 is well known from the literature. We remark that for a sufficiently well behaving payoff g, the unique maximum x_1^* is characterized by the condition

$$g(x_1^*)\psi_r'(x_1^*) = g'(x_1^*)\psi_r(x_1^*), \tag{5}$$

where, if necessary, the derivative g' is interpreted as directional from left. In the literature, the condition (5) is typically referred to as the *smooth pasting condition.* This refers to that if the condition (5) holds from both left and right, then the value is continuously differentiable at x_1^*.

3.2. **Timing problem #2: Restriction on admissible exercise times.** The second timing problem is a straightforward modification of the Problem (3). We impose an exogenous restriction on the decision makers ability to exercise. The restriction is set up using the independent Poisson process N described in Section 2 with $T_0 = 0$. We formulate the restriction as follows: the decision maker is allowed to stop only at the jump times of the Poisson process N. The optimization is now done over the set of $\mathbb{F}$-stopping times

$$\mathcal{T}_0 = \{\tau \ : \ \text{for all } \omega \in \Omega,\ \tau(\omega) = T_n(\omega) \text{ for some } n \in 0, 1, 2, \dots, \infty\}. \tag{6}$$

The optimal timing problem is formulated as

$$V_2(x) = \sup_{\tau \in \mathcal{T}_0} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau) \right]. \tag{7}$$

Next result characterizes the optimal exercise rule and the optimal value for Problem (7) under Assumption 3.1.

Theorem 3.3. *Let Assumption 3.1 hold. Then the threshold x_2^* characterized uniquely by the condition*

$$\psi_r(x_2^*) \int_{x_2^*}^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy = g(x_2^*) \int_{x_2^*}^{\infty} \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy$$

constitutes the optimal exercise rule for the optimal timing problem (7)*. Moreover, the value function $V_2 \in C(\mathbf{R}_+)$ can be rewritten as*

$$V_2(x) = \begin{cases} g(x), & x \geq x_2^*, \\ \frac{g(x_2^*)}{\psi_r(x_2^*)} \psi_r(x), & x < x_2^*. \end{cases} \tag{8}$$

Proof. See [21]. □

We observe from Theorem 3.3 that under Assumption 3.1, the functional forms of the values V_1 and V_2 are the same whereas the optimal exercise thresholds x_1^* and x_2^* differ. In contrast to V_1, we remark that V_2 is always non-differentiable over the optimal exercise boundary x_2^*.

3.3. **Timing problem #3: Exercise lag.** The second modification is concerned with another feature of the benchmark (3), namely immediate settlement of the exercise payoff. In contrast to Problem (7), the decision maker follows now the underlying diffusion and is free to choose the exercise time based on the observed information. However, at the time of the exercise, the exogenous Poisson process N is started and the payoff is settled at the first jump time of N. Equivalently, we can think that the exercise payoff is not determined by state of the underlying X the actual moment τ of the exercise but after an independent, exponentially distributed random lag $T_1 - T_0$ with mean λ^{-1} – now $T_0 = \tau$. The optimal timing problem with exercise lag is formulated as

$$V_3(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r(\tau+T_1)} g(X_{\tau+T_1}) \right], \tag{9}$$

where τ varies over all $\mathbb{G}$-stopping times. Next result gives a characterization of the optimal stopping rule and value function of Problem (9) under Assumption 3.1.

Theorem 3.4. *Let Assumption 3.1 hold. Then there is a unique threshold x_3^* characterized by the condition*

$$\frac{\int_0^{x_3^*} \psi_{r+\lambda}(y) g(y) m'(y) dy}{\int_0^{x_3^*} \psi_{r+\lambda}(y) \psi_r(y) m'(y) dy} = \frac{\int_{x_3^*}^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy}{\int_{x_3^*}^{\infty} \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy} \tag{10}$$

which constitutes the optimal exercise rule for the optimal timing problem (9). *Moreover, the value function $V_3 \in C^1(\mathbf{R}_+)$ can be rewritten as*

$$V_3(x) = \begin{cases} \lambda (R_{r+\lambda} g)(x), & x \geq x_3^*, \\ \frac{\lambda (R_{r+\lambda} g)(x_3^*)}{\psi_r(x_3^*)} \psi_r(x), & x < x_3^*. \end{cases} \tag{11}$$

To prove Theorem 3.4, we rewrite Problem (9) first as a perpetual problem with a suitably adjusted payoff function. First, let τ be an arbitrary $\mathbb{G}$-stopping time. Since the underlying X is strong Markov, the independence of the exponentially distributed lag variable $U := T_1 - T_0$ implies that

$$\mathbf{E}_x \left[e^{-r(\tau+T_1)} g(X_{\tau+T_1}) \right] = \mathbf{E}_x \left[e^{-r\tau} \mathbf{E}_{X_\tau} \left[e^{-rU} g(X_U) \right] \right] = \mathbf{E}_x \left[e^{-r\tau} \lambda (R_{r+\lambda} g)(X_\tau) \right],$$

for all $x \in \mathbf{R}_+$. Thus Problem (9) can be rewritten as

$$V_3(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau} \lambda (R_{r+\lambda} g)(X_\tau) \right], \tag{12}$$

where τ varies over all $\mathbb{G}$-stopping times. To study the solvability of Problem (12) under Assumption 3.1, we use the auxiliary functions $I_i : \mathbf{R}_+ \to \mathbf{R}$ and $J_i : \mathbf{R}_+ \to \mathbf{R}$, $i = 1, 2$, defined as

$$\begin{aligned} I_1(x) &= \int_x^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy, \quad I_2(x) = \int_0^x \psi_{r+\lambda}(y) g(y) m'(y) dy, \\ J_1(x) &= \int_x^{\infty} \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy, \quad J_2(x) = \int_0^x \psi_{r+\lambda}(y) \psi_r(y) m'(y) dy. \end{aligned} \tag{13}$$

We remark that the functions J_i are well-defined, since $\psi_r \in L_1^{r+\lambda}$ for all $\lambda > 0$. Indeed, since ψ_r is r-harmonic and nonnegative, we find using [21], Lemma 2.1, that

$$\mathbf{E}_x\left[\int_0^{\zeta} e^{-(r+\lambda)t}|\psi_r(X_t)|dt\right] = (R_{r+\lambda}\psi_r)(x) = \frac{1}{\lambda}\psi_r(x) < \infty,$$

for all $x \in \mathbf{R}_+$. Next result provides us with useful monotonicity properties of the functions I_i and J_i.

Lemma 3.5. *Let Assumption 3.1 hold. Then there are unique states $x_2^* < x_1^*$ and $\check{x} > x_1^*$ such that $x_2^* = \operatorname{argmax}\left\{\frac{I_1}{J_1}\right\}$ and $\check{x} = \operatorname{argmax}\left\{\frac{I_2}{J_2}\right\}$ and that the functions $x \mapsto \frac{I_1(x)}{J_1(x)}$ and $x \mapsto \frac{I_2(x)}{J_2(x)}$ are non-decreasing on $(0, x_2^*)$ and $(0, \check{x})$, and non-increasing on (x_2^*, ∞) and $(\check{x}, \infty)$, respectively.*

Proof. First, straightforward differentiation yields the condition

$$\frac{d}{dx}\left(\frac{I_1(x)}{J_1(x)}\right) \gtreqless 0 \text{ if and only if } \psi_r(x)I_1(x) \gtreqless g(x)J_1(x). \tag{14}$$

Let $x \geq x_1^*$. Since the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ is nonincreasing on (x_1^*, ∞), we find that

$$\begin{aligned}\psi_r(x)I_1(x) - g(x)J_1(x) &= \psi_r(x)\int_x^{\infty} \varphi_{r+\lambda}(y)\frac{g(y)}{\psi_r(y)}\psi_r(y)m'(y)dy - g(x)J_1(x) \\ &< \left(\psi_r(x)\frac{g(x)}{\psi_r(x)} - g(x)\right)J_1(x) = 0.\end{aligned}$$

Furthermore, since the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ tends to 0 as $x \to \infty$, we conclude using the condition (14) that the function $x \mapsto \frac{I_1(x)}{J_1(x)}$ is nonincreasing on $(x{*}_1, \infty)$ and tends to 0 as $x \to \infty$. On the other hand, since $\lim_{x\to 0+}\frac{g(x)}{\psi_r(x)} = 0$ and $\frac{I_1(\hat{x})}{J_1(\hat{x})} > 0$, we find using the condition (14) that the function $x \mapsto \frac{I_1(x)}{J_1(x)}$ must have at least one interior maximum $x_2^* < x_1^*$. Finally, since $\frac{g(x_2^*)}{\psi_r(x_2^*)} = \frac{I_1(x_2^*)}{J_1(x_2^*)}$, $x \mapsto \frac{I_1(x)}{J_1(x)}$ is continuously differentiable, and $x \mapsto \frac{g(x)}{\psi_r(x)}$ nondecreasing on $(0, x_1^*)$, we conclude, again using (14), that the maximum x_2^* is unique. The result for the function $x \mapsto \frac{I_2(x)}{J_2(x)}$ is proved completely analogously. □

Lemma 3.6. *Let Assumption 3.1 hold. Then there is a unique state $x_3^* = \operatorname{argmax}\left\{\frac{\lambda(R_{r+\lambda}g)(x)}{\psi_r(x)}\right\}$. Moreover, the function $x \mapsto \frac{\lambda(R_{r+\lambda}g)(x)}{\psi_r(x)}$ is non-decreasing on $(0, x_3^*)$ and non-increasing on (x_3^*, ∞).*

Proof. A straightforward application of first the harmonicity properties of the spanning functions $\psi_\cdot$ and $\varphi_\cdot$ and then Fundamental Theorem of Calculus yields

$$\begin{aligned}\frac{\psi'_{r+\lambda}(x)}{S'(x)}\psi_r(x) - \frac{\psi'_r(x)}{S'(x)}\psi_{r+\lambda}(x) &= \lambda\int_0^x \psi_{r+\lambda}(y)\psi_r(y)m'(y)dy, \\ \frac{\psi'_r(x)}{S'(x)}\varphi_{r+\lambda}(x) - \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\psi_r(x) &= \lambda\int_x^{\infty} \varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy.\end{aligned} \tag{15}$$

for all $x \in \mathbf{R}_+$. Recall the functions I_i and J_i, $i = 1, 2$, defined in (13). Using the formulas (15), we verify readily that

$$\frac{d}{dx}\left[\frac{\lambda(R_{r+\lambda}g)(x)}{\psi_r(x)}\right] = \frac{\lambda^2 S'(x)}{B_{r+\lambda}\psi_r^2(x)}\left(I_1(x)J_2(x) - I_2(x)J_1(x)\right) \gtreqless 0 \tag{16}$$
$$\text{if and only if } I_1(x)J_2(x) \gtreqless I_2(x)J_1(x).$$

First, recall that $x_2^* = \operatorname{argmax}\{\frac{I_1}{J_1}\}$ and let $x \in (0, x_2^*)$. Then Lemma 3.5 implies that the conditions $g(x)J_1(x) < \psi_r(x)I_1(x)$ and $g(x)J_2(x) > \psi_r(x)I_2(x)$ hold. Therefore

$$I_1(x)J_2(x) - I_2(x)J_1(x) > J_1(x)\left(\frac{g(x)}{\psi_r(x)}J_2(x) - I_2(x)\right) > 0.$$

Analogously we find that for all $x \in (\check{x}, \infty)$, where $\check{x} = \operatorname{argmax}\{\frac{I_2}{J_2}\}$,

$$I_1(x)J_2(x) - I_2(x)J_1(x) < J_2(x)\left(I_1(x) - \frac{g(x)}{\psi_r(x)}J_1(x)\right) < 0.$$

Thus, the function $x \mapsto \frac{\lambda(R_{r+\lambda}g)(x)}{\psi_r(x)}$ is increasing in $(0, x_2^*)$ and decreasing in $(\check{x}, \infty)$, and, by continuity, has a turning point $x_3^* \in (x_2^*, \check{x})$. Since the function $x \mapsto \frac{I_1(x)}{J_1(x)}$ is non-increasing and function $x \mapsto \frac{I_2(x)}{J_2(x)}$ is non-decreasing on $(x_2^*, \check{x})$, we conclude that turning point x_3^* must be unique. □

Proof of Theorem 3.4. In Lemma 3.6 we proved that under Assumption 3.1, the same assumptions are satisfied for the payoff function $x \mapsto \lambda(R_{r+\lambda}g)(x)$ of Problem (12) and threshold x_3^* instead of the original g and x_1^*. Thus, using [2], Theorem 3, we conclude that x_3^* constitutes the optimal exercise rule for Problem (9). Moreover, due to the smoothing effect of the Laplace transform $g \mapsto \lambda(R_{r+\lambda}g)$, the payoff $x \mapsto \lambda(R_{r+\lambda}g)(x)$, and, consequently, the value V_3 are continuously differentiable. Finally, the characterizing condition (10) is justified in (16). □

3.4. **Timing problem #4: Random time horizon.** The fourth problem is also of the classical form (3), but now the Poisson clock dictates the length of the time horizon. Analogously to Problem (7), both the underlying X and exogenous Poisson process N started with $T_0 = 0$. Moreover, she can choose the exercise time freely based on the observed information on the underlying. However, at the first jump time T_1, the opportunity to exercise expires. To study this setting, it is crucial whether the random time T_1 is an admissible stopping time. We make a distinction between these two cases and study first the case where the decision maker does not observe the jump – now T_1 is *not* an admissible stopping time. The optimal timing problem is now formulated as

$$V_4(x) = \mathbf{E}_x\left[e^{-r\tau}g(X_\tau)\mathbf{1}_{\{\tau<T_1\}}\right], \tag{17}$$

where τ varies over $\mathbb{G}$-stopping times. We remark that T_1 is not an $\mathbb{G}$-stopping time and that $\mathbf{P}(\tau = T_1) = 0$ for all $\mathbb{G}$-stopping times τ. Therefore it is irrelevant whether we write $\mathbf{1}_{\{\tau \leq T_1\}}$ or $\mathbf{1}_{\{\tau < T_1\}}$ in (17), see also [13]. The following result is our main result on the optimal exercise rule and the value function of Problem (17).

Theorem 3.7. *Let Assumptions 3.1 hold. Moreover, assume that there is a unique threshold* $x_4^* = \operatorname{argmax}\left\{\frac{g(x)}{\psi_{r+\lambda}(x)}\right\}$ *such that the function* $x \mapsto \frac{g(x)}{\psi_{r+\lambda}(x)}$ *is nondecreasing when* $x \leq x_4^*$ *and nonincreasing when* $x \geq x_4^*$*. Then* x_4^* *constitutes the optimal exercise rule for the optimal timing problem* (17)*. Moreover, the value function* V_4 *can be rewritten as*

$$V_4(x) = \begin{cases} g(x), & x \geq x_4^*, \\ \frac{g(x_4^*)}{\psi_{r+\lambda}(x_4^*)}\psi_{r+\lambda}(x), & x < x_4^*. \end{cases} \tag{18}$$

Following [13], we present the result which allows us to rewrite Problem (17) as an adjusted perpetual problem. The result follows from the independence of N and elementary properties of exponential distribution.

Proposition 3.8. *The value function* V_4 *can be expressed as*

$$V_4(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-(r+\lambda)\tau} g(X_\tau) \right],$$

where τ *varies over* $\mathbb{G}$*-stopping times.*

Proof of Theorem 3.7. Using Proposition 3.8, the claimed result follows from [2], Theorem 3. □

In Theorem 3.7 we showed that under the standing assumption 3.1 and an additional assumption of the shape of $x \mapsto \frac{g(x)}{\psi_{r+\lambda}(x)}$, the optimal timing problem (17) has a unique solution. We remark that from application point of view, this assumption does not add severe additional restriction. Informally it means that if the considered optimal timing problem has this particular type of solution for some discount rate $r > 0$, the form of the solution remains the same for an increased discount rate $r + \lambda$.

We turn now the case where the decision maker observes also the Poisson process N. Now, the jump time T_1 is an admissible stopping time and the optimal timing problem can be formulated as

$$V_5(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau) \mathbf{1}_{\{\tau \leq T_1\}} \right], \tag{19}$$

where τ varies over $\mathbb{F}$-stopping times. Next theorem gives is our main result on the optimal stopping rule and value function of Problem (19).

Theorem 3.9. *Assume that the payoff* g *satisfies the following*

(i) *$g \in L_1^r$ is non-decreasing with $g(0)=0$,*

(ii) *$g \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus D)$, where D is a finite subset of $\mathbf{R}_+$,*

(iii) *the limits $\lim_{x\to y\pm} g'(y) < \infty$ and $\lim_{x\to y\pm} g''(y) < \infty$ for all $y \in D$,*

(iv) *the function $(\mathcal{A}-r)g \in L_1^r$ and there is a unique state $\tilde{x}$ such that $(\mathcal{A}-r)g(x) \gtreqqless 0$ when $x \lesseqqgtr \tilde{x}$.*

Then the threshold x_5^* characterized uniquely by the condition

$$\int_0^{x_5^*} \psi_{r+\lambda}(y)(\mathcal{A}-r)g(y)m'(y)dy = 0 \tag{20}$$

constitutes the optimal exercise rule for the optimal timing problem (19)*. Moreover, the optimal value V_5 can rewritten as*

$$V_5(x) = \begin{cases} g(x), & x \geq x_5^*, \\ \lambda(R_{r+\lambda}g)(x) + \frac{g(x_5^*)-\lambda(R_{r+\lambda}g)(x_5^*)}{\psi_{r+\lambda}(x_5^*)}\psi_{r+\lambda}(x), & x < x_5^*. \end{cases} \tag{21}$$

Remark 3.10. *The assumptions of Theorem 3.9 are sufficient conditions for Assumption 3.1. Indeed, since $g(0)=0$, Corollary 3.2 of* [3] *implies that*

$$\frac{d}{dx}\left[\frac{g(x)}{\psi_r(x)}\right] = \frac{S'(x)}{\psi_r^2(x)} \int_0^x \psi_r(y)(\mathcal{A}-r)g(y)m'(y)dy.$$

Now property (iv) implies that there is a unique $x_1^* > \tilde{x}$ such that the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ is increasing on $(0, x_1^*)$ and decreasing on (x_1^*, ∞).

To prove Theorem 3.9, we first reformulate Problem (19) as a perpetual problem with a suitably adjusted payoff function. The following result is originally due to [13].

Theorem 3.11. *The value function V_5 can be expressed as*

$$V_5(x) = \sup_{\tau} \mathbf{E}_x \left[\lambda \int_0^{\tau} e^{-(r+\lambda)s} g(X_s)ds + e^{-(r+\lambda)\tau} g(X_\tau)\right],$$

where the supremum is taken over $\mathbb{G}$-stopping times.

Proof. Denote as $\mathcal{T}_0$ the set of all $\mathbb{G}$-stopping times, as $\mathcal{T}_1$ the set of all $\mathbb{F}$-stopping times, and as $\hat{\mathcal{T}}_1$ the set of all $\mathbb{F}$-stopping times τ, which satisfy $\tau \leq T_1$ for all ω. We know that for all $\tau \in \hat{\mathcal{T}}_1$, there is a $\tau' \in \mathcal{T}_1$ for which $\tau = \tau' \wedge T_1$. Using this, we find that

$$\begin{aligned} \sup_{\tau\in\mathcal{T}_1} \mathbf{E}_x\left[e^{-r\tau}g(X_\tau)\mathbf{1}_{\{\tau\leq T_1\}}\right] &= \sup_{\hat{\tau}\in\hat{\mathcal{T}}_1} \mathbf{E}_x\left[e^{-r\tau}g(X_\tau)\mathbf{1}_{\{\tau\leq T_1\}}\right] \\ &= \sup_{\hat{\tau}\in\hat{\mathcal{T}}_1} \mathbf{E}_x\left[e^{-r\tau}g(X_\tau)\right] = \sup_{\tau\in\mathcal{T}_1} \mathbf{E}_x\left[e^{-r(\tau\wedge T_1)}g(X_{\tau\wedge T_1})\right]. \end{aligned}$$

Now, it follows from [29], p. 370, that

$$\sup_{\tau\in\mathcal{T}_1}\mathbf{E}_x\left[e^{-r(\tau\wedge T_1)}g(X_{\tau\wedge T_1})\right]=\sup_{\tau\in\mathcal{T}_0}\mathbf{E}_x\left[e^{-r(\tau\wedge T_1)}g(X_{\tau\wedge T_1})\right].$$

Finally, let $\tau\in\mathcal{T}_0$. Then the independence of the Poisson process N implies

$$\begin{aligned}\mathbf{E}_x\left[e^{-r(\tau\wedge T_1)}g(X_{\tau\wedge T_1})\right]&=\mathbf{E}_x\left[e^{-r\tau}g(X_\tau)\mathbf{1}_{\{\tau\le T_1\}}\right]+\mathbf{E}_x\left[e^{-rT_1}g(X_{T_1})\mathbf{1}_{\{\tau>T_1\}}\right]\\&=\mathbf{E}_x\left[e^{-(r+\lambda)\tau}g(X_\tau)\right]+\lambda\mathbf{E}_x\left[\int_0^\tau e^{-(r+\lambda)s}g(X_s)ds\right],\end{aligned}$$

for all $x\in\mathbf{R}_+$. □

We proceed by using the alternate expression for V_5 given by Theorem 3.11. Since the underlying X is strong Markov, we find that

$$\begin{aligned}&\mathbf{E}_x\left[\lambda\int_0^\tau e^{-(r+\lambda)s}g(X_s)ds+e^{-(r+\lambda)\tau}g(X_\tau)\right]=\\&\mathbf{E}_x\left[\lambda\int_0^\zeta e^{-(r+\lambda)s}g(X_s)ds\right]-\mathbf{E}_x\left[\lambda\int_\tau^\zeta e^{-(r+\lambda)s}g(X_s)ds\right]+\mathbf{E}_x\left[e^{-(r+\lambda)\tau}g(X_\tau)\right]=\\&\lambda(R_{r+\lambda}g)(x)+\mathbf{E}_x\left[e^{-(r+\lambda)\tau}(g(X_\tau)-\lambda(R_{r+\lambda}g)(X_\tau))\right],\end{aligned}$$

for all $\mathbb{G}$-stopping times τ. Thus

$$V_5(x)=\lambda(R_{r+\lambda}g)(x)+\sup_\tau\mathbf{E}_x\left[e^{-(r+\lambda)\tau}(g(X_\tau)-\lambda(R_{r+\lambda}g)(X_\tau))\right],\tag{22}$$

where τ varies over all $\mathbb{G}$-stopping times. To study Problem (22), we have the following result.

Lemma 3.12. *Let the assumptions of Theorem 3.9 hold. Then there is a unique state* $x_5^*=\operatorname{argmax}\left\{\frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}\right\}$. *Moreover, the function* $x\mapsto\frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ *is nondecreasing on* $(0,x_5^*)$ *and nonincreasing on* (x_5^*,∞).

Proof. Using Corollary 3.2 from [3] we find that

$$\frac{d}{dx}\left[\frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}\right]=\frac{S'(x)}{\psi^2_{r+\lambda}(x)}\int_0^x\psi_{r+\lambda}(y)(\mathcal{A}-r)g(y)m'(y)dy,\tag{23}$$

for all $x\in\mathbf{R}_+$. First, since $(\mathcal{A}-r)g(x)>0$ on $(0,\tilde{x})$, we find that the function $x\mapsto\frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ is increasing on $(0,\tilde{x})$. On the other hand, thanks to Remark 3.10, there is a unique $x_1^*>\tilde{x}$ such that

$$\frac{d}{dx}\left[\frac{g(x)}{\psi_r(x)}\right]=\frac{S'(x)}{\psi_r^2(x)}\int_0^x\psi_r(y)(\mathcal{A}-r)g(y)m'(y)dy\gtreqless 0\text{ when }x\lesseqgtr x_1^*.\tag{24}$$

We note using (15) that the function $x \mapsto \frac{\psi_r(x)}{\psi_{r+\lambda}(x)}$ is decreasing. Since $\frac{g(x)}{\psi_{r+\lambda}(x)} = \frac{g(x)}{\psi_r(x)}\frac{\psi_r(x)}{\psi_{r+\lambda}(x)}$, we find that $x \mapsto \frac{g(x)}{\psi_{r+\lambda}(x)}$ is decreasing on (x_1^*, ∞). By coupling this with Lemma 3.5, we find that

$$\begin{aligned}\frac{d}{dx}\left[\frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}\right] &= g'(x)\psi_{r+\lambda}(x) - g(x)\psi'_{r+\lambda}(x) + \lambda S'(x)\int_0^x \psi_{r+\lambda}(y)g(y)m'(y)dy \\ &\leq \lambda\frac{S'(x)}{\psi_r(x)}\left(\psi_r(x)I_2(x) - g(x)J_2(x)\right) < 0,\end{aligned}$$

for all $x \in (x_1^*, \check{x})$. Finally, since $(\mathcal{A}-r)g(x) < 0$ when $x \geq \check{x}$, we conclude that $x \mapsto \frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ is decreasing also on $(\check{x}, \infty)$. Thus, the function $x \mapsto \frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ has a turning point $x_5^* \in (\tilde{x}, x_1^*)$, which is unique by the property (iv) of g and the condition (23). □

Proof of Theorem 3.9. Similarly to Lemma 3.6, we proved in Lemma 3.12 that under Assumption 3.1, the same assumptions are satisfied for the payoff $x \mapsto g(x) - \lambda(R_{r+\lambda}g)(x)$ and for the threshold x_5^*. The characterizing condition (20) is justified in (23) instead of the original g and x_1^*. Thus, using [2], Theorem 3 again, we conclude that Theorem 3.9 holds. □

In comparison to the previous timing problems, we posed more stringent smoothness assumption on the payoff function g in Theorem 3.9. In particular, we assumed that the payoff g is stochastically C^2, see, e.g, [11]. Moreover, we posed the assumption (iv) on the sub/superharmonicity of g. It appears to be difficult to relax these assumptions so that the remain convenient. Indeed, we saw in Lemma 3.12 that the function $x \mapsto \frac{g(x)-\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ is the key quantity in proving the existence of a unique optimal stopping threshold. However, for sufficiently large x, both $x \mapsto \frac{g(x)}{\psi_{r+\lambda}(x)}$ and $x \mapsto \frac{\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ turn out to be decreasing. To elaborate, recall the definitions from Lemma 3.5. It is a matter of straightforward differentiation to show that

$$\frac{d}{dx}\left[\frac{\lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}\right] = -\lambda\frac{S'(x)}{\psi_{r+\lambda}(x)}I_2(x) < 0$$

for all $x \in \mathbf{R}_+$. Moreover, we showed in the proof of Lemma 3.12 that $x \mapsto \frac{g(x)}{\psi_{r+\lambda}(x)}$ is decreasing on (x_1^*, ∞). Therefore, in order to prove Lemma 3.12, we should make a conclusion on the monotonicity of a difference of two decreasing functions on entire (x_1^*, ∞). This appears to be difficult without additional assumptions on the second order properties of g. However, there is still a wide variety of payoff functions, for example various piecewise linear payoffs, with economical and financial significance for which these more stringent assumptions hold.

It is worth pointing out that as opposed to optimal timing with deterministic finite horizon, see, e.g., [28], the optimal exercise boundary is in our case constant over time. This is an intuitive result, because the jump rate λ is constant. Indeed, even though the decision maker is aware of the random expiry of the opportunity, the expiry is conditionally equally probable on each equilong time interval. Therefore it is rational to keep a constant exercise threshold and add the rate λ as an increase in discounting. It is also worth highlighting that the value is expressed in (21) as a sum of the terminal value and the early exercise premium. In fact, we can interpret the term $\lambda(R_{r+\lambda}g)(x)$ as the expected present value of the exercise payoff if we exercise at the terminal time T_1. Thus, the remaining part $\frac{g(x_5^*)-\lambda(R_{r+\lambda}g)(x_5^*)}{\psi_{r+\lambda}(x_5^*)}\psi_{r+\lambda}(x)$ can be viewed as the value added by the possibility of exercising prior the expiry.

The models studied in this section are also studied in [13]. In comparison to [13], our analysis holds for much larger class of underlying processes X. In particular, we cover also case where the lower boundary of the state space is attainable to the state variable, i.e. X can hit 0 is finite time with positive probability – a feature which is desirable in many applications. Moreover, we emphasize different aspects of the model compared to [13]. In particular, we proposed general closed form expressions for the optimal values and exercise thresholds whereas in [13] the focus is on the connectivity of the waiting region.

3.5. **A Comparison of the models.** In the previous section we presented our main results characterizing the optimal exercise thresholds x_i^* and optimal values V_i, $i = 1, \dots, 5$, under Assumptions 3.1. In this section we study the properties of the optimal characteristics as functions of the rate λ. Intuitively it seems clear that in Problems (7) and (9) the values should tend to the value V_1 as λ increases. Indeed, increased λ should result into shorter expected gaps between the admissible exercise times in Problem (7) and in shorter expected lag after the exercise in Problem (9). Using the same intuition, Problems (17) and (19) appear to be qualitatively different from the other two. For these problem, we reason that increased λ results into shorter expected time horizon, which should lower the value. Following proposition shows that our models concur with this intuition.

Proposition 3.13. *The value functions V_i, $i = 2, \dots, 5$, and the corresponding optimal thresholds satisfy the limiting properties*

- $V_2 \to V_1$, $V_3 \to V_1$, *and* $V_5 \to g$ *as* $\lambda \to \infty$,
- $V_2 \to g$, $V_3 \to 0$, $V_4 \to V_1$, *and* $V_5 \to V_1$ *as* $\lambda \to 0$,
- *thresholds* x_3^* *and* x_2^* *tend to* x_1^* *as* $\lambda \to \infty$,
- *thresholds* x_4^* *and* x_5^* *tend to* x_1^* *as* $\lambda \to 0$.

Proof. The claimed limiting properties of V_2 and x_2^* are proved in [21]. To proceed, we note that under our assumptions the payoff g is bounded on $\mathbf{R}_+$. Therefore an elementary modification of [20], Lemma 3.1.2, p. 65, yields the pointwise convergence $\lambda(R_{r+\lambda}g)(x) \to g(x)$ as $\lambda \to \infty$ for all $x \in \mathbf{R}_+$. Moreover, since $g \in L_1^r$, we find that $\lambda(R_{r+\lambda}g)(x) \to 0$ as $\lambda \to 0$ for all $x \in \mathbf{R}_+$. Given these limiting properties and the expression

(11), we find that the claimed limiting properties of V_3 hold. Similarly, we find from expression (21) that the claimed limiting properties of V_5 hold. The limiting property of V_4 follows immediately from Theorem 3.7. Finally, given the convergence results of value functions V_3, V_4 and V_5, the claimed convergence results hold also for thresholds x_3^*, x_4^* and x_5^*. □

In addition to asymptotic properties of the value functions V_i, Proposition 3.13 provides us also with information on the limiting behavior of optimal exercise thresholds x_i^*, $i = 2, \ldots, 5$. In the next proposition, we elaborate the limiting results on these thresholds.

Proposition 3.14. *For any fixed $\lambda > 0$, the optimal stopping thresholds x_i^*, $i = 1, \ldots, 5$ satisfy the orderings*

$$x_2^* \leq x_1^*, \quad x_2^* \leq x_3^*, \quad x_4^* \leq x_5^* \leq x_1^*. \tag{25}$$

Proof. The first inequality is proved in [21]. The second inequality is established in the proof of Lemma 3.5. The inequalities $x_4^* \leq x_1^*$ and $x_4^* \leq x_1^*$ are established in the proof of Lemma 3.12. Finally, since g is non-negative, we find that $(\mathcal{A} - (r + \lambda))g(x) \leq (\mathcal{A} - r)g(x)$ for all $x \in \mathbf{R}_+$ and, consequently, thanks to the expression (23) and Remark 3.10, that $x \mapsto \frac{g(x) - \lambda(R_{r+\lambda}g)(x)}{\psi_{r+\lambda}(x)}$ is increasing on $(0, x_4^*)$. □

Proposition 3.14 states an interesting but intuitive result. Indeed, it shows unambiguously that in Problems (7), (17), and (19) the introduced Poissonian time uncertainty accelerates the optimal exercise, i.e., a rational agent will lower the return requirement of the investment project. In Problem (7) the decision maker decides at every jump time of the Poisson process N whether to exercise or not. In addition to this this, she is also aware that if she decides to wait, she is exposed to the risk of losing a profitable moment. Therefore it is natural that she will "play it safe" in the sense that she lowers her return requirement. Similarly, in Problems (17) and (19), if the decision maker decides to wait, there is a risk that the underlying starts to decrease and the opportunity expires with a low level of return. To compensate this, the decision maker lowers the exercise threshold.

It is interesting to note from Proposition 3.14 that for a fixed rate λ, the optimal exercise threshold is lowered less in the presence of the exercise lag than in the presence of the restriction on the admissible exercise times. Recall the example from the introduction where the goal was to time optimally a disinvestment option. We remarked that the restriction on the admissible exercise times corresponds to the case where the market is irresponsive to the supply generated by the decision makers attempt to realize the production machinery. This is in contrast to the case of exercise lag, where exercise triggers the exponential waiting time to have the demand and, consequently, to realize. In this sense, the exogenous Poissonian constraint is not as restricting in the latter case as it is in the first, which is reflected in the model as an increased return requirement.

As a final remark, we note that at least in principle, the considered time uncertainties can be bundled into a single timing problem. For example, we could consider the combination of restriction in exercise times and exercise lag. However, in order to apply the results the this study, we have to assume that the each uncertainty is dictated by different Poisson processes, which are mutually independent.

4. An Illustration

In this section we illustrate the main results of the study with a classical example familiar, for example, from [23]. We assume the underlying dynamics X follow a geometric Brownian motion, i.e., the regular linear diffusion X given as the solution of the Itô equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$. Here, W is a Wiener process. The scale density S' reads as $S'(x) = x^{-\frac{2\mu}{\sigma^2}}$ and the speed density m' reads as $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}}$. It is well known that in this case the differential operator $\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}$ and that the minimal excessive functions $\psi.$ and $\varphi.$ can be written as

$$\begin{cases} \psi_r(x) = x^b, \\ \varphi_r(x) = x^a, \end{cases} \qquad \begin{cases} \psi_{r+\lambda}(x) = x^\beta, \\ \varphi_{r+\lambda}(x) = x^\alpha, \end{cases}$$

where the constants

$$\begin{cases} b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \\ a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0, \end{cases} \qquad \begin{cases} \beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1, \\ \alpha = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0. \end{cases}$$

It is a simple computation to show that the Wronskian $B_{r+\lambda} = 2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$.

We consider the payoff $g(x) = (x-K)^+$ for a fixed constant $K > 0$. Then Problem (3) can be written as

$$V_1(x) = \sup_\tau \mathbf{E}_x \left[e^{-r\tau}(X_\tau - K)^+ \right]. \tag{26}$$

For the sake of finiteness, we assume that $\mu < r$ and $\mu - \frac{1}{2}\sigma^2 > 0$. This guarantees that we have the optimal exercise thresholds are finite and are attained almost surely in a finite time. We note that in this case the assumptions of Theorem 3.9 hold and that the function $x \mapsto \frac{g(x)}{\psi_{r+\lambda}(x)}$ satisfies the condition required in Theorem 3.7. This allows us to compute the optimal characteristics of all considered timing problems. In particular, we find that

$$(\mathcal{A}-r)g(x) = \begin{cases} (\mu - r)x + rK, & x > K, \\ 0, & x \leq K, \end{cases}$$

and that on $x > K$, the function $x \mapsto (\mathcal{A}-r)g(x)$ has a unique zero at $\tilde{x} = \frac{rK}{r-\mu}$.

For the fixed model specification, it is well known that the optimal exercise rules for Problems (3) and (17) are constituted by the thresholds

$$x_1^* = \frac{bK}{b-1} > K, \quad x_4^* = \frac{\beta K}{\beta - 1} > K,$$

and that the value functions V_1 and V_4 can be written as

$$V_1(x) = \begin{cases} x - K, & x \geq x_1^*, \\ \frac{x_1^* - K}{x_1^{*b}} x^b, & x < x_1^*, \end{cases} \quad V_4(x) = \begin{cases} x - K, & x \geq x_4^*, \\ \frac{x_4^* - K}{x_4^{*\beta}} x^\beta, & x < x_4^*, \end{cases}$$

see, e.g., [23]. For Problem (7) with restricted admissible exercise times, it is proved in [17], see also [21], that the optimal exercise threshold x_2^* reads as

$$x_2^* = \frac{b(\beta-1)}{\beta(b-1)} K = \frac{b - \frac{r}{r+\lambda} a}{b - \frac{(r-\mu)a - \lambda}{r+\lambda-\mu}} K < x_1^*,$$

and the value function V_2 can be written as

$$V_2(x) = \begin{cases} x - K, & x \geq x_2^*, \\ \frac{x_2^* - K}{x_2^{*b}} x^b, & x < x_2^*. \end{cases}$$

We turn now to Problem (9), i.e., to the problem with exercise lag. Since the payoff $g(x) = (x-K)^+ = 0$ when $x \leq K$, we find after straightforward integration that the resolvent $\lambda(R_{r+\lambda} g)$ can be written as

$$\lambda(R_{r+\lambda} g)(x) = \begin{cases} \frac{\lambda}{r+\lambda-\mu} x - \frac{\lambda}{r+\lambda} K - \frac{2\lambda K^{1-\alpha}}{\sigma^2 B_{r+\lambda} \alpha(1-\alpha)} x^\alpha, & x > K, \\ \frac{2\lambda K^{1-\beta}}{\sigma^2 B_{r+\lambda}(\beta-1)\beta} x^\beta, & x \leq K. \end{cases}$$

To determine the optimal exercise threshold x_3^*, recall first the definition (13). Again, it is a matter of straightforward integration to establish that

$$\begin{cases} I_1(x) = \begin{cases} \frac{2}{\sigma^2}x^{-\beta}\left(\frac{x}{\beta-1}-\frac{K}{\beta}\right) & x > K, \\ \frac{2K^{-(\beta-1)}}{\sigma^2\beta(\beta-1)} & x < K, \end{cases} \\ J_1(x) = \frac{2}{\sigma^2\kappa}x^{-\kappa}, \end{cases} \qquad \begin{cases} I_2(x) = \begin{cases} \frac{2}{\sigma^2(1-\alpha)\alpha}\left(x^{-\alpha}(K+\alpha(x-K))-K^{1-\alpha}\right) & x > K, \\ 0 & x < K, \end{cases} \\ J_2(x) = \frac{2}{\sigma^2\gamma}x^{\gamma}, \end{cases}$$

where

$$\begin{cases} \kappa = \sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2(r+\lambda)}{\sigma^2}} - \sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}, \\ \gamma = \sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2(r+\lambda)}{\sigma^2}} + \sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}. \end{cases}$$

Using these expressions, we verify readily that

$$\begin{aligned} \frac{I_1(x)}{J_1(x)} &= \frac{\kappa x^{-b}}{\beta(\beta-1)}\left(\beta x - K(\beta-1)\right), \\ \frac{I_2(x)}{J_2(x)} &= \frac{\gamma x^{-\gamma}}{\alpha(1-\alpha)}\left(x^{-\alpha}(K+\alpha(x-K))-K^{1-\alpha}\right), \end{aligned}$$

for all $x \in \mathbf{R}_+$. From these equations, we can solve the optimal exercise threshold x_3^* numerically.

To illustrate Problem (19), we find the optimal exercise threshold $x_5^* > K$ can be identified as the unique root of the equation

$${x_5^*}^{\beta}\mathbf{1}_{\{x_5^*>K\}} - \beta {x_5^*}^{\beta-1}(x_5^*-K)^+ + \lambda {x_5^*}^{-\frac{2\mu}{\sigma^2}}\Phi(x_5^*) = 0,$$

where

$$\Phi(x) = \begin{cases} \frac{2}{\sigma^2}\left(\frac{x^{1-\alpha}-K^{1-\alpha}}{1-\alpha}+K\frac{x^{-\alpha}-K^{-\alpha}}{\alpha}\right), & x \geq K, \\ 0, & x < K. \end{cases}$$

From this condition, we can compute the optimal exercise threshold x_5^*.

To illustrate the results on asymptotics, the numerical values of the optimal exercise thresholds x_i^*, $i = 1, \ldots, 4$, are presented in Table 1 for various values of the parameter λ. In particular, the parameter configuration is fixed as $r = 0.05$, $\mu = 0.025$, $\sigma = 0.15$, $K = 2$. For this configuration, the threshold $x_1^* = 5.425$.

We observe that the numerical results listed in Table 1 are in line with our main results. First, the thresholds x_i^*, $i = 2, \ldots, 5$, appear to converge to x_1^* as was indicated by Proposition 3.13. Moreover, the threshold x_1^* dominates thresholds x_2^*, x_4^* and x_5^*. We also note that the convergence of x_3^* to x_1^* is from below, so for this example the Poissonian time uncertainty accelerates the optimal exercise also in this case. Moreover, the numerics concur with the result that the threshold x_3^* giving rise to the optimal decision rule

λ	0.01	0.1	1	10	100
x_2^*	2.374	3.670	4.827	5.240	5.368
x_3^*	2.879	4.480	5.296	5.412	5.424
x_4^*	4.571	2.956	2.248	2.071	2.022
x_5^*	5.374	4.964	4.391	4.131	4.042

Table 1. The optimal exercise thresholds x_i^* for various values of λ under the parameter configuration $\mu = 0.025$, $r = 0.05$, $\sigma = 0.15$, and $K = 2$.

under exercise lag dominates the threshold x_2^* associated to the problem where admissible exercise times are restricted to the jump times of Poisson process N. We also observe that both of these thresholds are increasing as functions of the rate λ, whereas the thresholds x_4^* and x_5^* of the random time horizon problems are decreasing. We remark that difference between x_4^* and x_5^* becomes significant for large values λ. This highlights the significance of the one additional admissible exercise time T_1 to the optimal decision rule. Interestingly, it appears that threshold $x_4^* \to 4$ for this parameter configuration as λ increases. This can be explained using the proof of Lemma 3.12. Indeed, we observe from the proof that threshold x_4^* dominates always the state $\tilde{x}$ where the payoff becomes r-superharmonic - for the current parameters, we find that $\tilde{x} = 4$.

To close the section, graphical illustrations of the value functions V_i, $i = 1, \dots, 4$, are presented in Figures 1, 2, and 3. Moreover, the relative distances $\frac{V_i}{V_1}$ are presented in Figure 4. The parameter configuration is now fixed as $r = 0.05$, $\mu = 0.0175$, $\sigma = 0.175$, $\lambda = 0.1$ and $K = 1.2$.

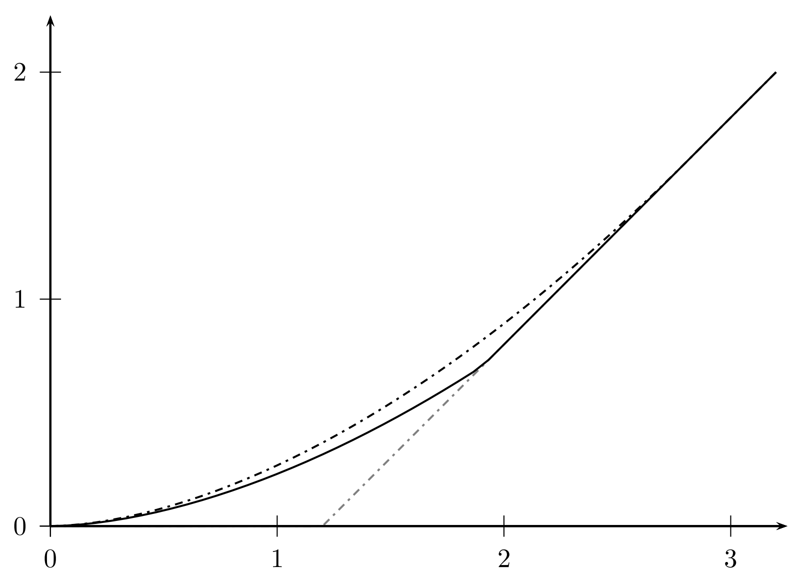

Figure 1. Values V_1 (black dashed curve) and V_2 (black solid curve) and the payoff $g : x \mapsto (x - K)^+$ (grey dashed curve) under the parameter configuration $r = 0.05$, $\mu = 0.0175$, $\sigma = 0.175$, $\lambda = 0.1$ and $K = 1.2$. The corresponding exercise thresholds are $x_1^* = 2.828$ and $x_2^* = 1.904$

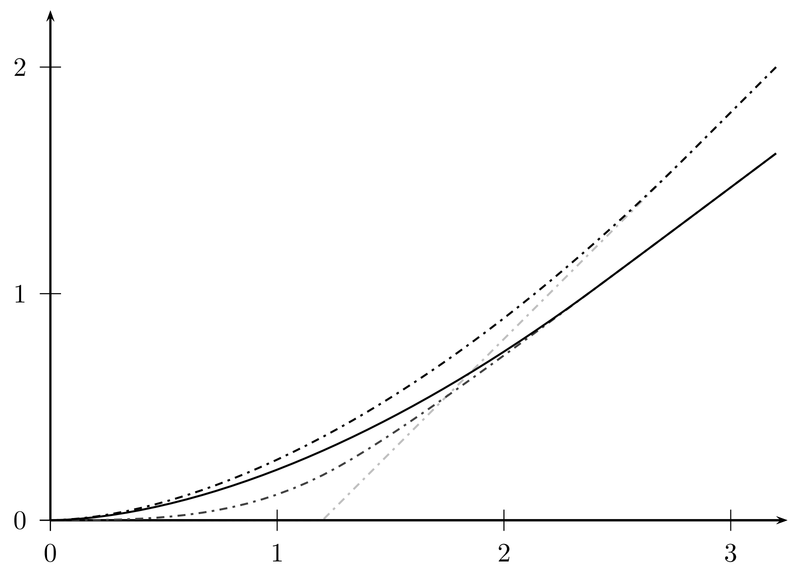

Figure 2. Values V_1 (black dashed curve) and V_3 (black solid curve) and the payoffs $g : x \mapsto (x - K)^+$ (light grey dashed curve) and $x \mapsto \lambda(R_{r+\lambda}g)(x)$ (dark grey dashed curve) under the parameter configuration $r = 0.05$, $\mu = 0.0175$, $\sigma = 0.175$, $\lambda = 0.1$ and $K = 1.2$. The corresponding exercise thresholds are $x_1^* = 2.828$ and $x_3^* = 2.410$

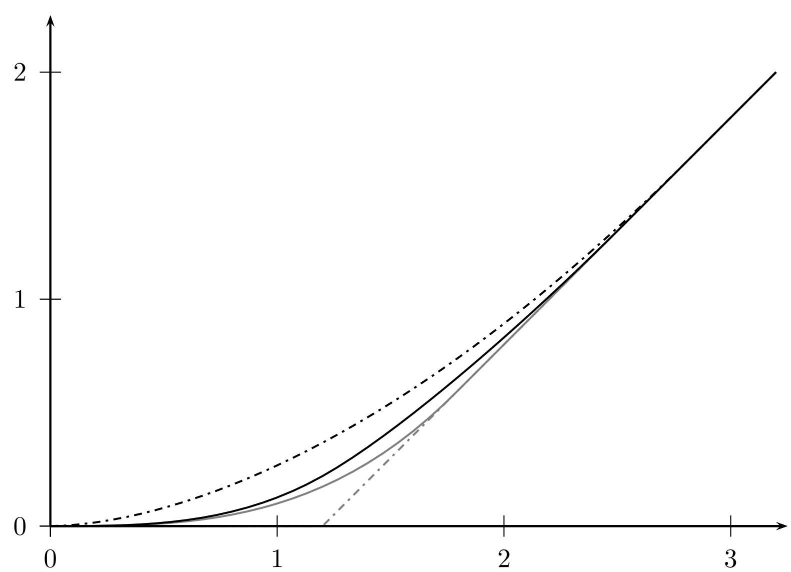

Figure 3. Values V_1 (black dashed curve), V_4 (grey solid curve) and V_5 (black solid curve) and the payoff $g : x \mapsto (x - K)^+$ (grey dashed curve) under the parameter configuration $r = 0.05$, $\mu = 0.0175$, $\sigma = 0.175$, $\lambda = 0.1$ and $K = 1.2$. The corresponding exercise thresholds are $x_1^* = 2.828$ and $x_4^* = 2.529$

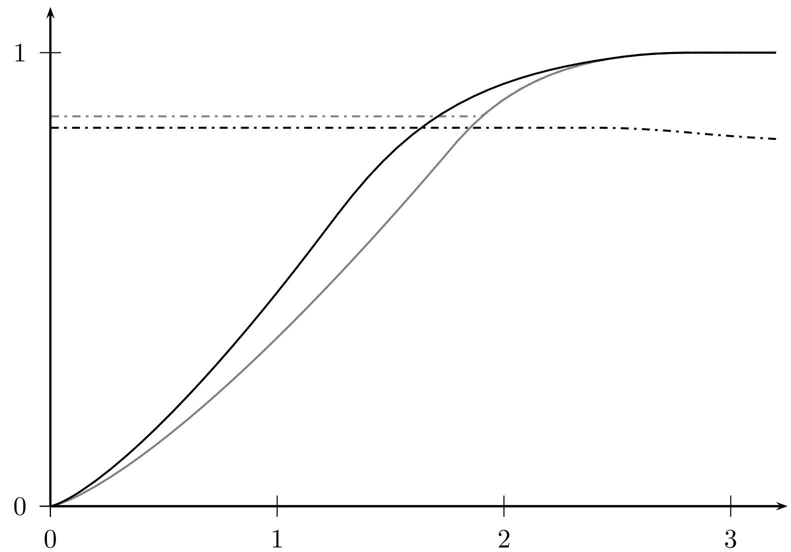

Figure 4. The relative distances $\frac{V_i}{V_1}$, $i = 2, \ldots, 5$, (grey dashed line, black dashed line, black solid line and grey solid line, respectively) under the parameter configuration $r = 0.05$, $\mu = 0.0175$, $\sigma = 0.175$, $\lambda = 0.1$ and $K = 1.2$

We find from Figures 1-3 that the graphics are in line with our main results. In particular, we find that the value V_2 is continuous over the threshold x_2^* and the value V_3 is smooth over the threshold x_3^*. Figure 4 highlights the qualitative difference of the problems with random time horizon to the other considered timing problems. In particular, we find that for small initial values, the relative distances $\frac{V_4}{V_1}$ and $\frac{V_5}{V_1}$ approach 1. Thus there is a chance of very severe overvaluation if a classical perpetual model is used in the case where there is actually a random time horizon. Moreover, if the classical model is used for valuation in a setting with exercise lag, the opportunity is overvalued for all initial states; for the used parameters $\frac{V_3(x)}{V_1(x)} < 0.84$ for all $x \in \mathbf{R}_+$. In terms of relative distance, the smallest overvaluation is done using the classical model with respect to Problem (7). For this problem, we find that $\frac{V_2(x)}{V_1(x)} > 0.85$ for all $x \in \mathbf{R}_+$ for the used parameter configuration, which can result into a severe overvaluation, especially on the absolute scale.

5. Concluding Comments

In this paper, we studied optimal timing of an irreversible investment decision under return and time uncertainty. As a benchmark we used the classical perpetual optimal stopping problem á-la Samuelson-McKean. We proposed and studied three other optimal timing problems, into which we incorporated an independent Poisson process. In first of these problem, exercising was allowed only at the jump times of the Poisson process. The second problem contained an independent, exponentially distributed exercise lag and the third a random time horizon with the same characteristics. Moreover, we studied two different versions of the random time horizon problem.

We stated and proved that the first three problems can be solved under relatively weak standing assumptions 3.1, which are quite easy to check for particular examples, at least numerically. Moreover, we proposed a set of more stringent assumptions for the random time horizon problems under which we proved the solvability of the problems. We also showed that under these assumptions, which are again quite easy to check numerically and relevant from applications point of view, all considered problems are solvable. This enabled us to compare to optimal characteristics of the timing models. In particular, we established that for restricted exercise times and random time horizon, the Poissonian time uncertainty accelerates the optimal exercise, i.e., the optimal return requirement is lowered. We also observed that for a fixed rate λ, the return requirement is lowered less for the problem with exercise lag than for the problem where exercise is allowed only at the jump times of the Poisson process.

We considered in this paper the case where the rate λ is constant over time. It would be interesting to see if some of the results of this study could generalized to case where λ is given a dynamical structure. This question is left for future research.

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